# Infinitesimal Phase Response Curves and Adjoint Variational Equation

Grisha Bordyugov, gregbg@gmail.com

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## 1 Introduction

In this document, I am trying to convince the reader about a systematic approach to introduce Infinitesimal Phase Response Curves (iPRCs) for small perturbations. This approach accounts for perturbations along arbitrary directions and components of the limit cycle. The iPRCs are essentially given by the solution of the adjoint variational equation around the unperturbed limit cycle and hence can be computed in a systematic manner, for instance, by standard continuation software like AUTO, MatCont, etc, or by a direct numerical integration of the adjoint equation in the backward time.

Here, I deliberately introduced the term "*infinitesimal*" to distinguish them apart from "*normal*" PRCs which sometime are considered to be dependent on the direction on the applied perturbation.

## 2 Setup

I consider a limit cycle oscillator for a variable $u \in \mathbb{R}^N$ in the form of the ODE:

$$u' = f(u; p) \tag{1}$$

with some parameters $p \in \mathbb{R}^P$. Let $u(t) = s(t)$ is a $T$-periodic limit cycle (i.e. $s(t+T) = s(t)$ for all $t$) in Eq. (1), i.e. it satisfies $s' = f(s; p)$.

## 2.1 Linearization about limit cycle

For a small perturbation $v(t)$ around the limit cycle $s(t)$ we have a linerized equation

$$v' = A(t)v, \tag{2}$$

where the time-dependent Jacobian matrix is given by $A(t) = f_u(s(t);p)$ Denote by $\Phi(s,t)$ the evolution operator, corresponding to that linear equation, such that the solutions of Eq. (2) can be written as

$$v(t) = \Phi(t,0)v(0). \tag{3}$$

Note that the inverse of the evolution operator $\Phi(s,t)$ is given by $\Phi^{-1}(s,t) = \Phi(t,s)$.

**The Goldstone mode and phase shifts of the limit cycle** One solution of Eq. (2) is given by the derivative of the limit cycle itself, that is by $s'(t)$. To see that, differentiate $s' = f(s;p)$ with respect to time $t$, obtaining

$$s'' = f_u(s;p)s',$$

which is nothing else as Eq. (2) with $v = s'(t)$.

The interpretation of this is the following: A small phase shift of the limit cycle $s(t)$ is written down as $s(t + \delta t)$. Expanding $s(t + \delta t)$ in Taylor series in $\delta t$ to the first order results in

$$s(t + \delta t) = s(t) + s'(t)\delta t + \mathcal{O}(\delta t^2).$$

We see that a phase shift is equivalent to adding a $\delta t$-small multiple of the Goldstone mode $s'(t)$ to the unshifted limit cycle $s(t)$.

**The eigen-thingy approach** A bit more abstractly, we could try to solve Eq. (2) by the Ansatz $v(t) = q(t)\,\mathrm{e}^{\lambda t}$ with a scalar $\lambda$ and a function $q(t)$. This results in

$$v'(t) = \left(q(t)\,\mathrm{e}^{\lambda t}\right)' = \left(q'(t) + \lambda q(t)\right)\mathrm{e}^{\lambda t}.$$

Substituting this result in Eq. (2), we obtain

$$\left(q'(t) + \lambda q(t)\right)\mathrm{e}^{\lambda t} = A(t)q(t)\,\mathrm{e}^{\lambda t} \quad \Rightarrow \quad q'(t) + \lambda q(t) = A(t)q(t) \quad \Rightarrow \quad \underbrace{\left(A(t) - \partial_t\right)}_{=\mathcal{L}}q(t) = \lambda q(t).$$

This can be considered as an eigenvalue problem for the operator

$$\mathcal{L} = A(t) - \partial_t$$

with the eigenvalue $\lambda$ and the eigenfunction $q(t)$. In this formulation, the derivative of the limit cycle $s'(t)$ is the eigenfunction to $\mathcal{L}$ to the eigenvalue $\lambda = 0$:

$$\mathcal{L}s'(t) = 0.$$

## 2.2 Adjoint equation

The adjoint (see Section 4 for a formal definition of an adjoint operator) to Eq. (2) is given by

$$v' = -A^+(t)v \tag{4}$$

The evolution operator to Eq. (4) is given by the adjoint of the inverse of the evolution operator $\Phi(s,t)$ to Eq. (2), i.e. by $\left(\Phi^{-1}(s,t)\right)^+$. Equation (4) can be reformulated as the eigenvalue problem

$$\mathcal{L}^+v = \lambda v$$

for the adjoint operator given by

$$\mathcal{L}^+ = A^+(t) + \partial_t.$$

**The adjoint Goldstone mode**  The adjoint $\mathcal{L}^+$ also has a zero eigenvalue, and we denote the corresponding eigenfunction by $\psi(t)$, i.e.

$$\mathcal{L}^+ \psi(t) = 0,$$

or equivalently

$$\psi_t(t) = -A^+(t)\,\psi(t).$$

Note that the above equation is just a simple linear ODE with the time-periodic coefficient matrix $-A^+(t)$. It can be solved using the standard ODE solvers.

# 3   Adjoint eigenfunctions as iPRCs

## 3.1   Perturbed limit cycle

Consider a perturbed limit cycles oscillator

$$u' = f(u; p) + g, \tag{5}$$

where $g = g(t)$ is a generic, but small perturbation. We are now interested in the phase difference between $s(t)$ (the unperturbed limit cycle) and the solution of Eq. (5) after one period $T$.

## 3.2   Linearization

In order to solve Eq. (5), we make an ansatz

$$u(t) = s(t) + w(t) \tag{6}$$

with small $w(t)$. We also assume that $w(0) = 0$, i.e. the unperturbed and the perturbed solutions share the same initial condition.

Substituting Ansatz (6) into Eq. (5) and expanding the nonlinearity in Taylor series to the first order, we obtain

$$s' + w' = f(s; p) + f_u(s; p)\,w + g.$$

Since $s$ satisfies $s' = f(s; p)$, we are left with the simple linear inhomogeneous ODE

$$w' = f_u(s(t); p)\,w + g \qquad \Rightarrow \qquad w' = A(t)\,w + g,$$

recall the definition of the matrix $A(t)$ as the Jacobian $f_u(s(t); p)$

## 3.3   Solving the perturbed equation

The solution to the perturbed (inhomogeneous) linear equation

$$w' = A(t)\,w + g \tag{7}$$

is given by the variation of constants formula

$$w(t) = \Phi(t, 0)\,w(0) + \int_0^t \Phi(t, s)\,g(s)\,\mathrm{d}s, \tag{8}$$

where $\Phi(s,t)$ is the evolution operator of the corresponding homegenous problem $v' = A(t)v$, see Eq. (3). Indeed a straight-forward differentiation results in[1]

$$w'(t) = \left(\Phi(t,0)w(0) + \int_0^t \Phi(t,s)\,g(s)\,\mathrm{d}s\right)' = A(t)w + \Phi(t,t)g(t) + \int_0^t \frac{\partial}{\partial t}\left(\Phi(t,s)g(s)\right)\,\mathrm{d}s =$$

$$= A(t)\,\Phi(t,0)\,w(0) + g(t) + \int_0^t A(t)\,\Phi(t,s)\,g(s)\,\mathrm{d}s =$$

$$= A(t)\left(\Phi(t,0)w(0) + \int_0^t \Phi(t,s)\,g(s)\,\mathrm{d}s\right) + g(t) = A(t)\,w(t) + g(t),$$

which is exactly the right-hand side of Eq. (7).

## 3.4 Main result: Phase difference introduced by perturbation

We are now interested in the phase jump $\Delta$ after one period $T$, which is produced by adding the perturbation $g$ in Eq. (5. We aim to compute $\Delta$ as the difference between the perturbed solution $u(t)$ and the unperturbed limit cycle $s(t)$ after one period $T$, that is $u(T) - s(T) = w(T)$ projected on the direction tangent to the limit cycle, i.e. spanned by $s'(0)$. This projection is given by the scalar product with the adjoint eigenfunction (the adjoint Goldstone mode) $\psi$:

$$\Delta = \langle \psi(0), w(T)\rangle.$$

From Eq. (8) we have that

$$w(T) = \Phi(T,0)w(0) + \int_0^T \Phi(T,s)\,g(s)\,\mathrm{d}s = \int_0^T \Phi(T,s)\,g(s)\,\mathrm{d}s,$$

since we assumed $w(0) = 0$. We hence obtain for the phase difference

$$\Delta = \langle \psi(0), w(T)\rangle = \left\langle \psi(0), \int_0^T \Phi(T,s)\,g(s),\mathrm{d}s\right\rangle =$$

$$= \int_0^T \langle \psi(0), \Phi(T,s)\,g(s)\rangle\,\mathrm{d}s = \int_0^T \langle \Phi^+(s,T)\,\psi(0), g(s)\rangle\,\mathrm{d}s = \int_0^T \langle \Phi^+(s,0)\,\psi(0), g(s)\rangle\,\mathrm{d}s =$$

$$= \int_0^T \langle \psi(s), g(s)\rangle\,\mathrm{d}s.$$

Here we used that by periodicity $\psi(T) = \psi(0)$ and that $(\Phi^+(s,t))^{-1} = \Phi^+(t,s)$, i.e. that the inverse of the evolution operator is the evolution in the reversed time.

That's basically our main result that relates the amount of the phase jump $\Delta$ to the external perturbation $g$:

$$\Delta = \int_0^T \langle \psi(s), g(s)\rangle\,\mathrm{d}s. \tag{9}$$

In the case of a $\delta$-perturbation $g(t) = g_0\,\delta(t - t_0)$ with a constant vector $g_0 \in \mathbb{R}^N$ which specifies the direction of the perturbation, we obtain the classical PRC

$$\Delta(t_0) = \langle \psi(t_0), g\rangle,$$

which expresses the amount of phase jump $\Delta$ in dependence on the moment $t_0$ of perturbation.

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[1]See, for instance, `http://en.wikipedia.org/wiki/Differentiation_under_the_integral_sign` for how to differentiate the integral.

# 4    Technical

## 4.1    Vector space and scalar product

In the following, we consider $a, b$ to be elements of a vector space $X$ with a scalar product

$$\langle a, b \rangle.$$

Examples include $N$-dimensional vector spaces $\mathbb{R}^N$ with the standard scalar product. Another example is a space of $L$-periodic functions $a(x), b(x)$ with the scalar product

$$\langle a, b \rangle = \int\limits_0^L \mathrm{d}x \, a(x) \, b(x)$$

## 4.2    Adjoint operator

For a given operator $\mathcal{A}$, its adjoint is $\mathcal{A}^+$ such that for any $a, b$

$$\langle a, \mathcal{A}b \rangle = \langle \mathcal{A}^+ a, b \rangle.$$

Note that the adjoint depends on the choice of the space and, in particular, on the choice of the scalar product $\langle \cdot, \cdot \rangle$.

Small exercise: prove that the adjoint of an $N \times N$ matrix is just its complex-conjugate transpose.

Other examples: With periodic boundary conditions $a(T) = a(0)$, the adjoint of the first derivative is minus first derivative, and second derivative is self-adjoint:

$$\langle u, v_x \rangle = -\langle u_x, v \rangle,$$
$$\langle u, v_{xx} \rangle = \langle u_{xx}, v \rangle.$$

(Check it yourself using the integration by parts rule!)

The adjoint to the linearization around a limit cycle

$$v_t = f_u(u) \, v$$

is hence given by

$$v_t = -f_u^+(u) \, v,$$

where the plus superscript means the transpose of the Jacobian $f_u$.

## 4.3    Eigenvectors of the $\mathcal{A}^+$ are orthogonal to those of $\mathcal{A}$

Let $v_j$ and $\lambda_j$ be eigenvectors of $\mathcal{A}$ and $\psi_i$ and $\mu_i$ those of $\mathcal{A}^+$, i.e.:

$$\mathcal{A} v_j = \lambda_j v_j,$$
$$\mathcal{A}^+ \psi_i = \mu_i \psi_i$$

Let us compute the scalar product

$$\langle \psi_i, \mathcal{A} v_j \rangle$$

for some $i$ and $j$ which must be not necessarily equal. On one hand we have

$$\langle \psi_i, \mathcal{A} v_j \rangle = \langle \psi_i, \lambda_j v_j \rangle = \lambda_j \langle \psi_i, v_j \rangle,$$

and on the other

$$\langle \psi_i, \mathcal{A} v_j \rangle = \langle \mathcal{A}^+ \psi_i, v_j \rangle = \langle \mu_i \psi_i, v_j \rangle = \mu_i^* \langle \psi_i, v_j \rangle,$$

where the star denotes the complex-conjugate of $\mu_i$ [2]. Note that the complex conjugate is applicable only to complex-valued eigenvalues, which we do not directly need for the iPRC, but I leave it in the complex form form for the generality. Comparing the last two equations, we see

$$\lambda_j \langle \psi_i, v_j \rangle = \mu_i^* \langle \psi_i, v_j \rangle \qquad \Rightarrow \qquad (\mu_i^* - \lambda_j) \langle \psi_i, v_j \rangle = 0 \qquad \Rightarrow \qquad \begin{cases} \mu_i^* = \lambda_j, & \langle \psi_i, v_j \rangle \neq 0 \\ \mu_i^* \neq \lambda_j, & \langle \psi_i, v_j \rangle = 0. \end{cases}$$

The last equation tells us that

1. $\mathcal{A}$ and $\mathcal{A}^+$ possess complex conjugate eigenvalues $\mu_i^* = \lambda_j$

2. the eigenvectors to different eigenvalues, i.e. $\mu_i^* \neq \lambda_j$, are orthogonal to each other, i.e. $\langle \psi_i, v_j \rangle = 0$.

We can always choose the labeling $i$ and $j$ in such a way that $\mu_i^* = \lambda_i$. Then we have

$$\langle \psi_i, v_j \rangle = 0, \quad \text{if} \quad i \neq j. \tag{10}$$

This is the desired **orthogonality relation** between the eigenvectors of $\mathcal{A}$ and $\mathcal{A}^+$.

## 4.4 What is it useful for?

### 4.4.1 For projection on an eigenvector of a given matrix

Suppose that $\mathcal{A}$ is a matrix with $N$ eigenvectors $v_i, i = 1, 2, \ldots, N$, and we would like to decompose a given vector in the basis of $v_i$, i.e. for a given vector $a$ find numbers $\alpha_i$ such that

$$a = \alpha_1 v_1 + \alpha_1 v_2 + \cdots + \alpha_N v_N. \tag{11}$$

**Self-adjoint case** If all $v_i$ *were* orthogonal to each other: $\langle v_i, v_j \rangle = 0$ for $i \neq j$ (which is not the general case), we could form a scalar product of Eq. (11) with $v_i$ to find $\alpha_i$:

$$\langle v_i, a \rangle = \alpha_1 \langle v_i, v_1 \rangle + \alpha_1 \langle v_i, v_2 \rangle + \cdots + \alpha_N \langle v_i, v_N \rangle.$$

Due to the assumed orthogonality $\langle v_i, v_j \rangle = 0$ for $i \neq j$, all scalar products on the right-hand side will be killed except for $\langle v_i, v_i \rangle$. We then find $\alpha_i$ immediately from

$$\langle v_i, a \rangle = \alpha_i \langle v_i, v_i \rangle \qquad \Rightarrow \qquad \alpha_i = \frac{\langle v_i, a \rangle}{\langle v_i, v_i \rangle}.$$

**General case** For a general matrix $\mathcal{A}$, its eigenvectors are not orthogonal to each other, i.e. the condition $\langle v_i, v_j \rangle = 0$ for $i \neq j$ does not hold. Instead, we can solve the problem of finding $\alpha_i$ by forming a scalar product of Eq. (11) with $\psi_i$ being an eigenvector of the adjoint $\mathcal{A}^+$:

$$\langle \psi_i, a \rangle = \alpha_1 \langle \psi_i, v_1 \rangle + \alpha_1 \langle \psi_i, v_2 \rangle + \cdots + \alpha_N \langle \psi_i, v_N \rangle.$$

Using Eq. (10), we see that all scalar products in the last equation will be killed except for

$$\langle \psi_i, a \rangle = \alpha_i \langle \psi_i, v_i \rangle,$$

and hence we can compute the desired coefficient $\alpha_i$:

$$\alpha_i = \frac{\langle \psi_i, a \rangle}{\langle \psi_i, v_i \rangle}.$$

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[2] Recall that for a complex number $\alpha$, the scalar product must satisfy $\langle a, \alpha b \rangle = \alpha \langle a, b \rangle$ and $\langle \alpha a, b \rangle = \alpha^* \langle a, b \rangle$.

### 4.4.2 Range of $\mathcal{A}$ and the nullspace of $\mathcal{A}^+$

... still needs to be completed ...

The idea is that the range of the operator $\mathcal{A}$ (i.e. all possible $\mathcal{A}a$ for any $a$) is orthogonal to the nullspace of the adjoint $\mathcal{A}^+$. Suppose that $b$ is from the nullspace of $\mathcal{A}^+$, that is by the definition of the nullspace

$$\mathcal{A}^+ b = 0,$$

than for any $a$ we have

$$\langle b, \mathcal{A}a \rangle = \langle \mathcal{A}^+ b, a \rangle = \langle 0, a \rangle = 0.$$

Application to the linear stability analysis of limit cycles: Given a limit cycle $u(\phi)$ as the solution to equation

$$u' = f(u),$$

the derivative $v = u'$ is the Goldstone mode, i.e. solves the linearized problem

$$v' = f_u(u) v.$$

The adjoint equation

$$\omega v_t = -f_u^+(u) v$$

also has a solution, which we denoted by $\psi$. Moreover, we have the orthogonality

$$\langle \psi, (\omega \partial_t - f_u(u)) v \rangle = 0$$

for any $v$ and can normalize the adjoint solution $\psi$ (as a solution of a linear equation, $\psi$ is defined up to a constant) in such a way that

$$\langle \psi, u' \rangle = 1.$$