Last lecture overview

- Many phenomena are spatially extended: Structures, spatio-temporal patterns
- Need PDEs to model - more involving mathematics and numerics
- Spatial structures with characteristic length scale can spontaneously emerge even in spatially homogeneous environments, example: Turing instability

Turing instability

two interacting and diffusing species

$$u_t = f(u, v) + D_u u_{xx},$$
$$v_t = g(u, v) + D_v v_{xx}$$

homogeneous steady state (HSS)

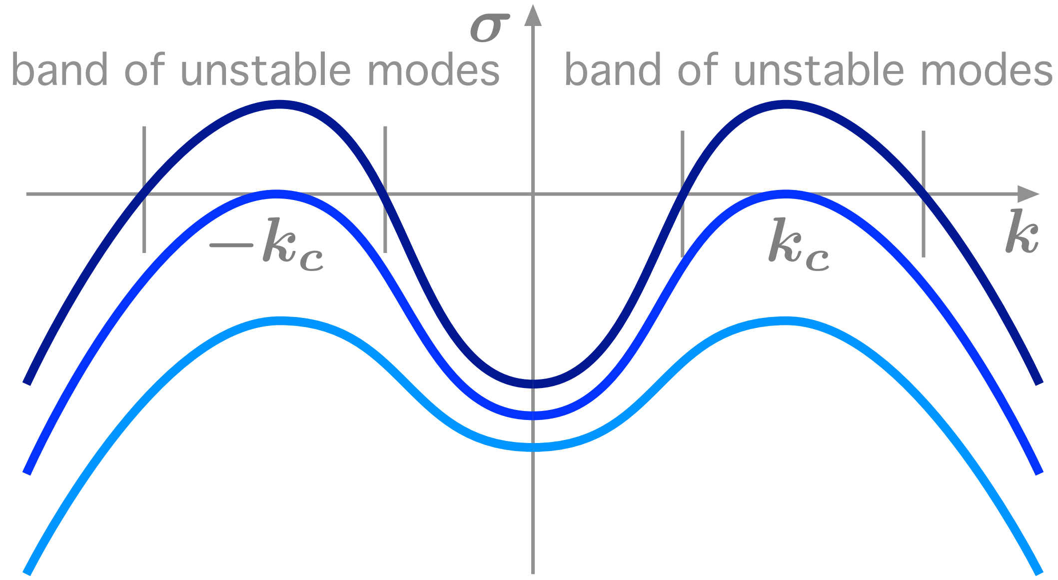
$$u = 0, v = 0$$

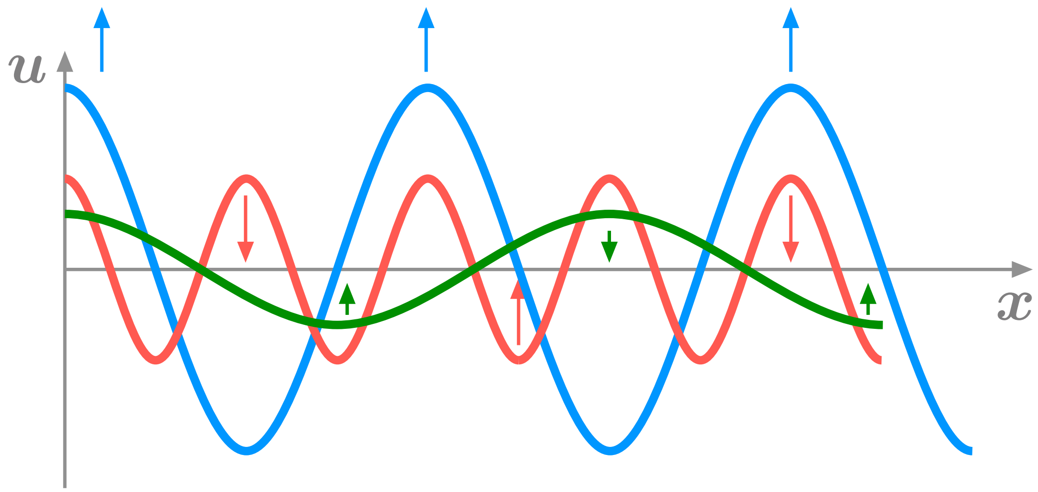
perturbation about HSS

$$u(x, t) = 0 + u_0 \mathrm{e}^{\mathrm{i}kx + \sigma t},$$
$$v(x, t) = 0 + v_0 \mathrm{e}^{\mathrm{i}kx + \sigma t}$$

dispersion:

$$d(k, \sigma) = 0$$





Turing instability: Length scales

linearized eqs

$$u_t = au - bv + D_u u_{xx},$$
$$v_t = cu - dv + D_v v_{xx}$$

$$a, d \propto 1/\mathrm{time},$$
$$D_u, D_v \propto \mathrm{length}^2/\mathrm{time}$$

$$\frac{a}{D_u}, \frac{d}{D_v} \propto 1/\mathrm{length}^2$$

critical wave number

$$k_c^2 = \frac{1}{2}\left(\frac{a}{D_u} - \frac{d}{D_v}\right)$$

$$k_c^2 > 0 \quad \Rightarrow \quad \frac{a}{D_u} > \frac{d}{D_v}$$

diffusion lengths

$$l_u = \sqrt{\frac{D_u}{a}}, \quad l_v = \sqrt{\frac{D_v}{d}}$$

$$\frac{D_u}{a} < \frac{D_v}{d} \quad \Rightarrow \quad l_u < l_v$$

Turing instability: Activator-Inhibitor interaction

linearized eqs

$$u_t = au - bv + D_u u_{xx},$$
$$v_t = cu - dv + D_v v_{xx}$$

$$a, d > 0$$

locally:
u activates itself, whereas
v inhibits itself:
activator-inhibitor system

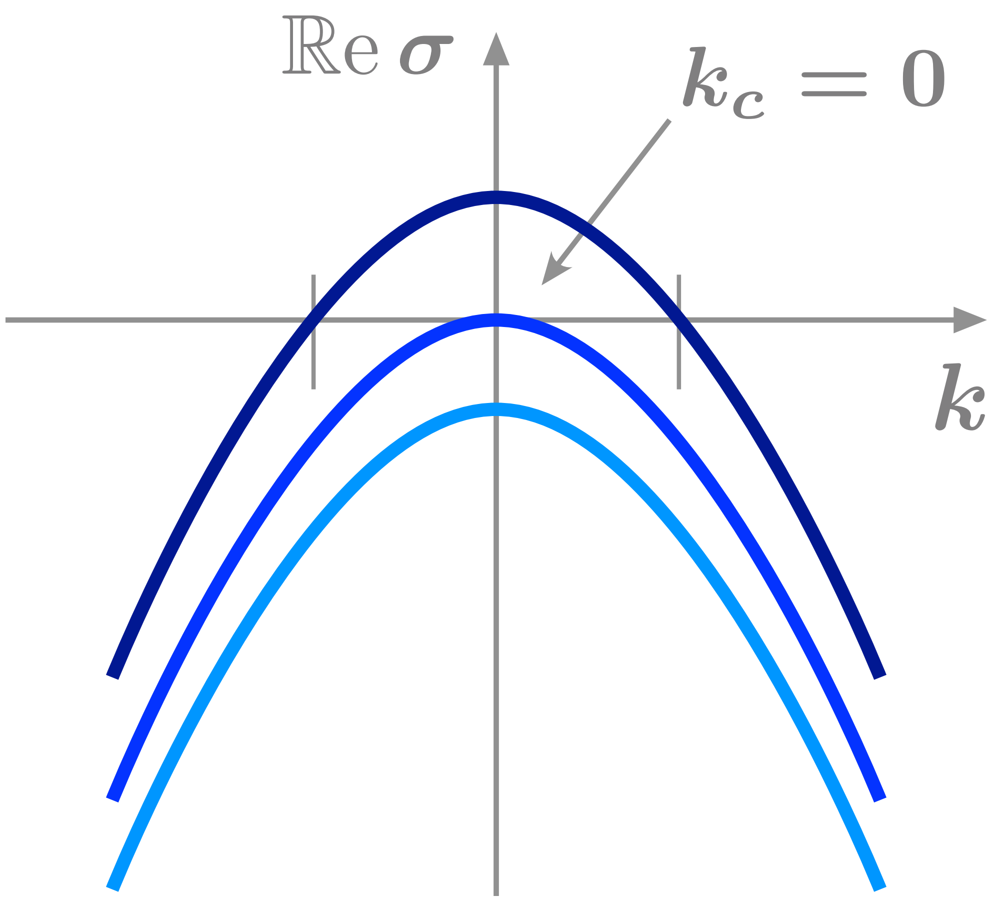
$$\frac{D_u}{a} < \frac{D_v}{d} \quad \Rightarrow \quad l_u < l_v$$

activator has a smaller diffusion length:

short-range activation
+
long range inhibition

Due to the difference in the diff coeffs, experimentalist had hard time finding a laboratory example of the Turing instability. Solution: immobilize activator in a gel, thus decreasing its diffusion length

Homogeneous Hopf instability



$$\mathbb{Re}\,\sigma_c = 0, \quad k_c = 0, \quad \mathbb{Im}\,\sigma_c \neq 0$$

Example:

$$z_t = \left((\lambda + \mathrm{i}\omega) - |z|^2\right) z + z_{xx}$$

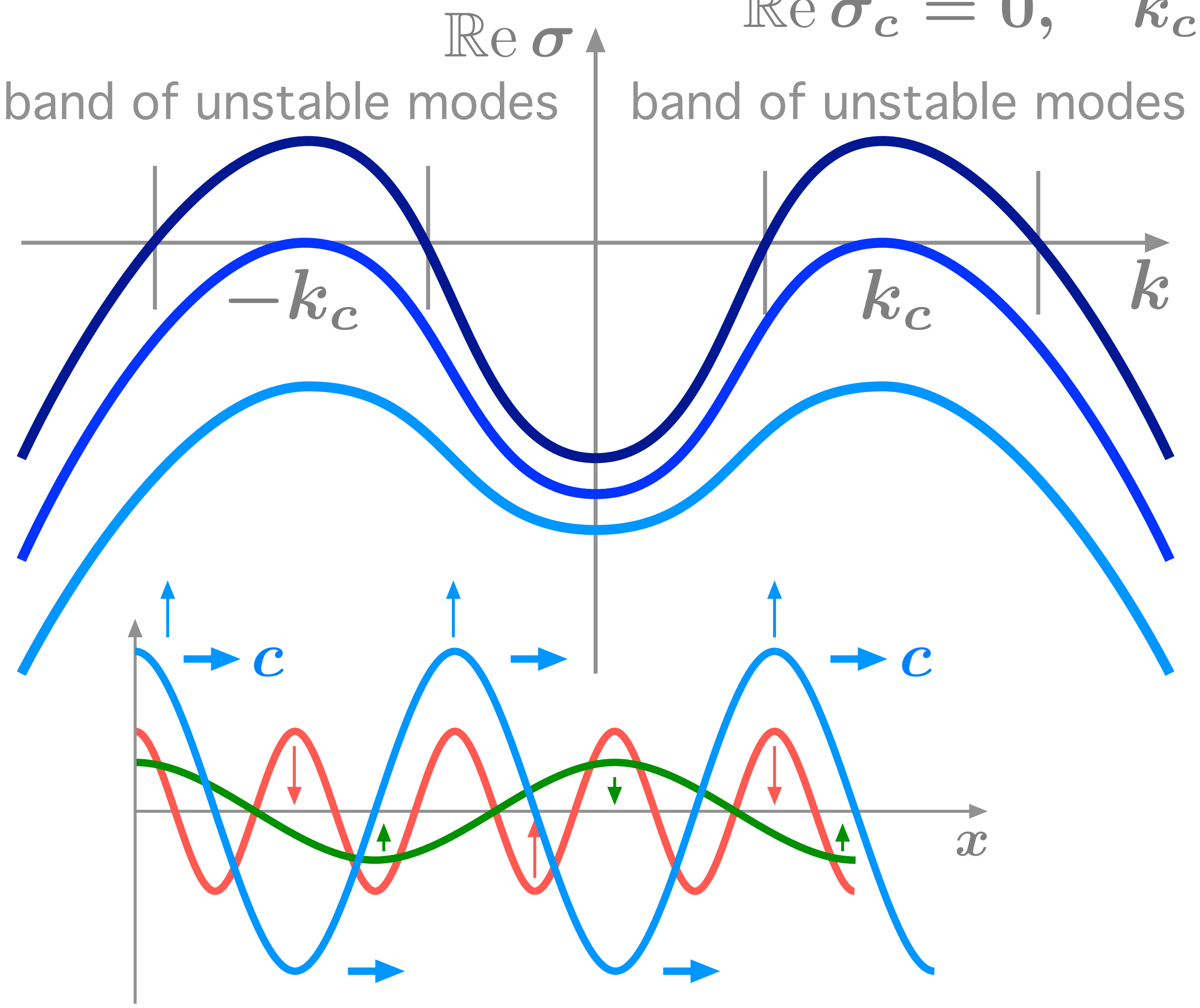
dispersion:

$$\sigma = \lambda + \mathrm{i}\omega - k^2$$

emergence of spatially homogeneous oscillations

Turing-Hopf (a.k.a. wave) instability

$$\mathbb{Re}\,\sigma_c = 0, \quad k_c \neq 0, \quad \mathbb{Im}\,\sigma_c \neq 0$$



emergence of propagating waves with velocity

$$c \approx \frac{\mathbb{Im}\,\sigma_c}{k_c}$$

Instabilities of HSS

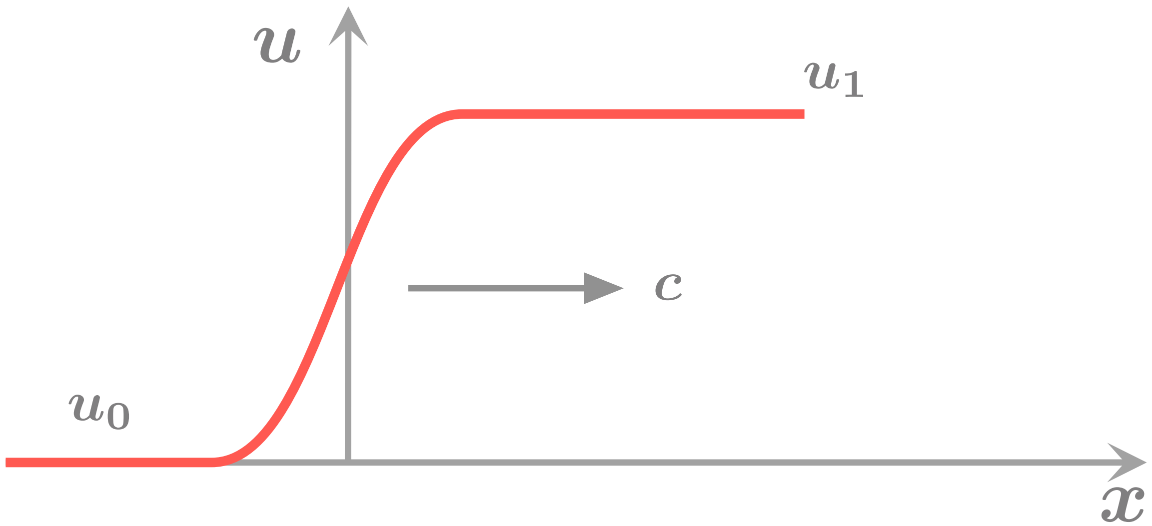
	k_c	$\mathrm{Im}\,\sigma_c$	emerging pattern
Turing	$k_c \neq 0$	$\mathrm{Im}\,\sigma_c = 0$	stationary wave
Hopf	$k_c = 0$	$\mathrm{Im}\,\sigma_c \neq 0$	homogeneous oscillation
Turing-Hopf (wave)	$k_c \neq 0$	$\mathrm{Im}\,\sigma_c \neq 0$	running wave with speed

Fronts in bistable RDSs

$$u_t = f(u) + u_{xx}$$

two stable HSSs

$$f(u_0) = f(u_1) = 0$$



wave of transition between
two otherwise stable states

propagation of fire fronts

Fronts in bistable RDSs

$$u_t = f(u) + u_{xx}$$

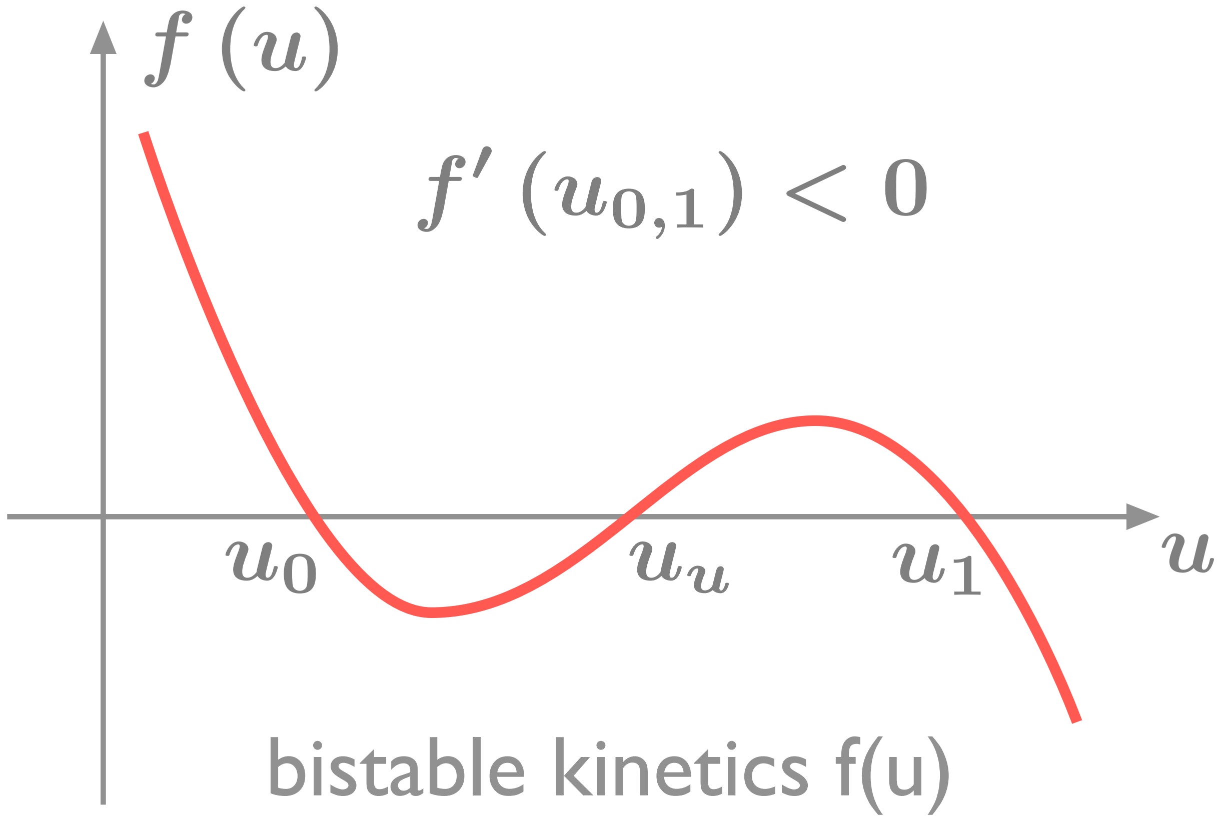
moving coordinate

$$z = x - ct$$

partial derivatives:

$$\partial_t u \underbrace{(x - ct)}_{z} = u_z \cdot z_t = -cu_z$$

$$\partial_{xx} u \underbrace{(x - ct)}_{z} = u_{zz} \cdot (z_x)^2 = u_{zz}$$



bistable kinetics f(u)

$$-cu_z = f(u) + u_{zz} \quad \Rightarrow \quad u_{zz} + cu_z + f(u) = 0$$

equation for a particle moving in force field -f(u)!

z is our new "time"

Fronts in bistable RDSs

$$u_{zz} + cu_z + f(u) = 0$$

Newton's 2nd law:

$$u_{zz} = -cu_z - f(u)$$

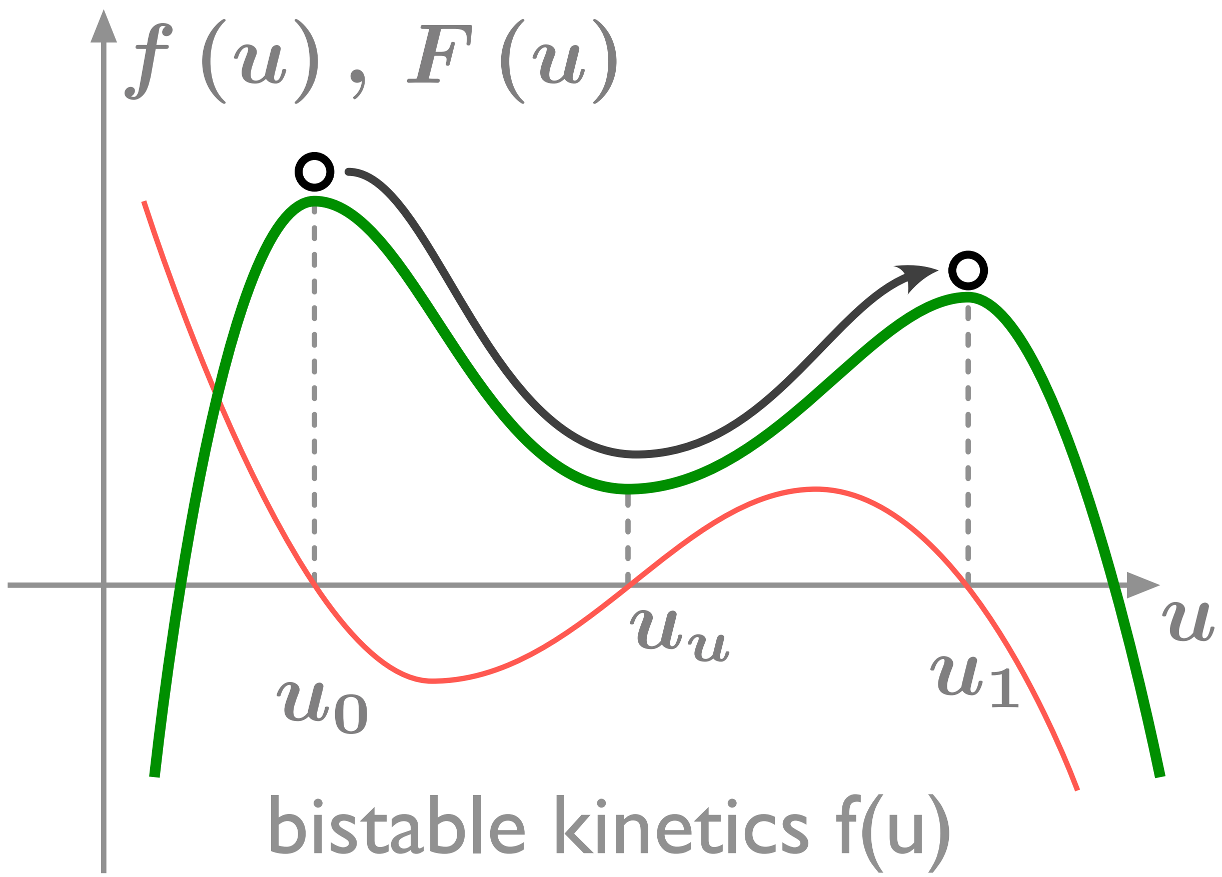
potential F(u) $F'(u) = f(u)$

c - friction coeff

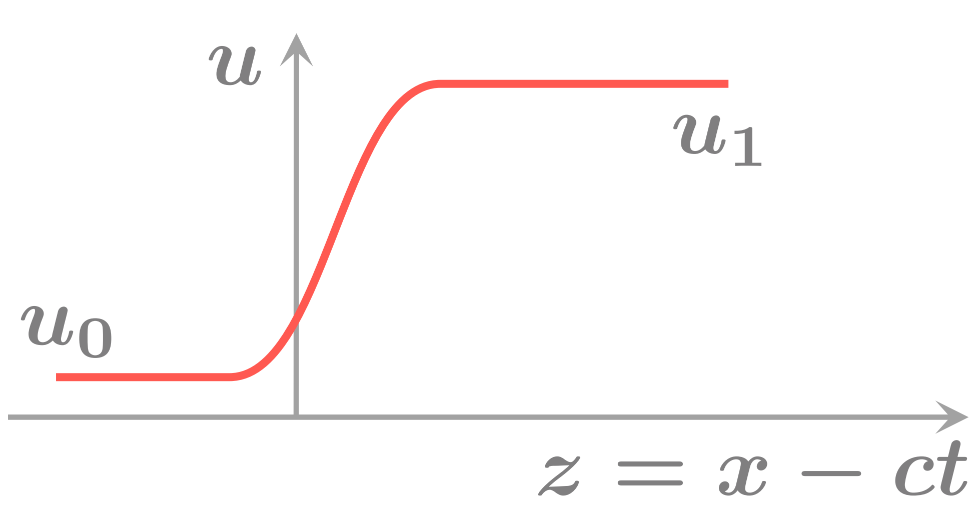
boundary conditions:

$$u(z = -\infty) = u_0,$$
$$u(z = \infty) = u_1$$

can be satisfied with just one value of friction c !!!



bistable kinetics f(u) and potential F(u)

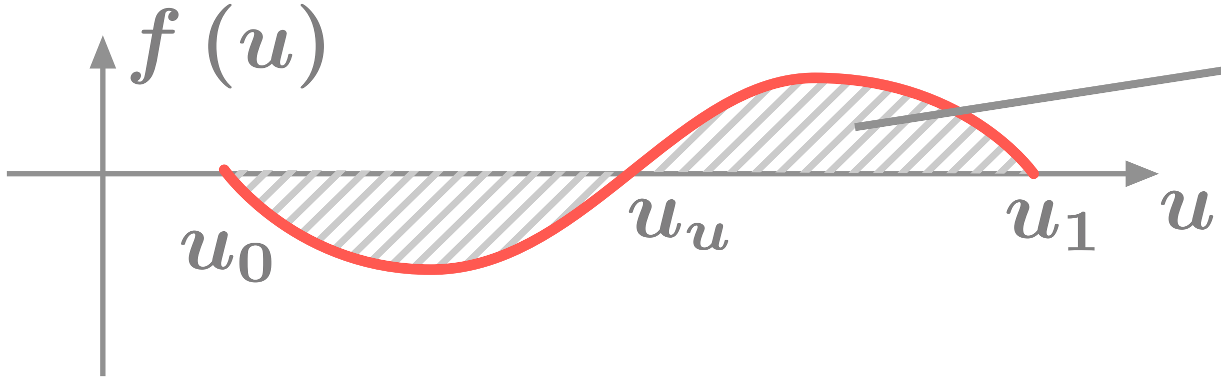


Fronts in bistable RDSs

$$u_{zz} + c u_z + f(u) = 0 \qquad \Big| \times u_z$$

$$u_{zz} u_z + c\,(u_z)^2 + f(u)\,u_z = 0 \qquad \Big| \int_{-\infty}^{\infty} \mathrm{d}z$$

$$\underbrace{\int_{-\infty}^{\infty} \mathrm{d}z\, u_{zz} u_z}_{=0} + \int_{-\infty}^{\infty} \mathrm{d}z\, c\,(u_z)^2 + \int_{-\infty}^{\infty} \mathrm{d}z\, f(u)\, u_z = 0 \qquad \Rightarrow$$



$$c = - \frac{\int_{u0}^{u1} \mathrm{d}u\, f(u)}{\int_{-\infty}^{\infty} \mathrm{d}z\, (u_z)^2}$$

Excitable media

"La Ola" Wave

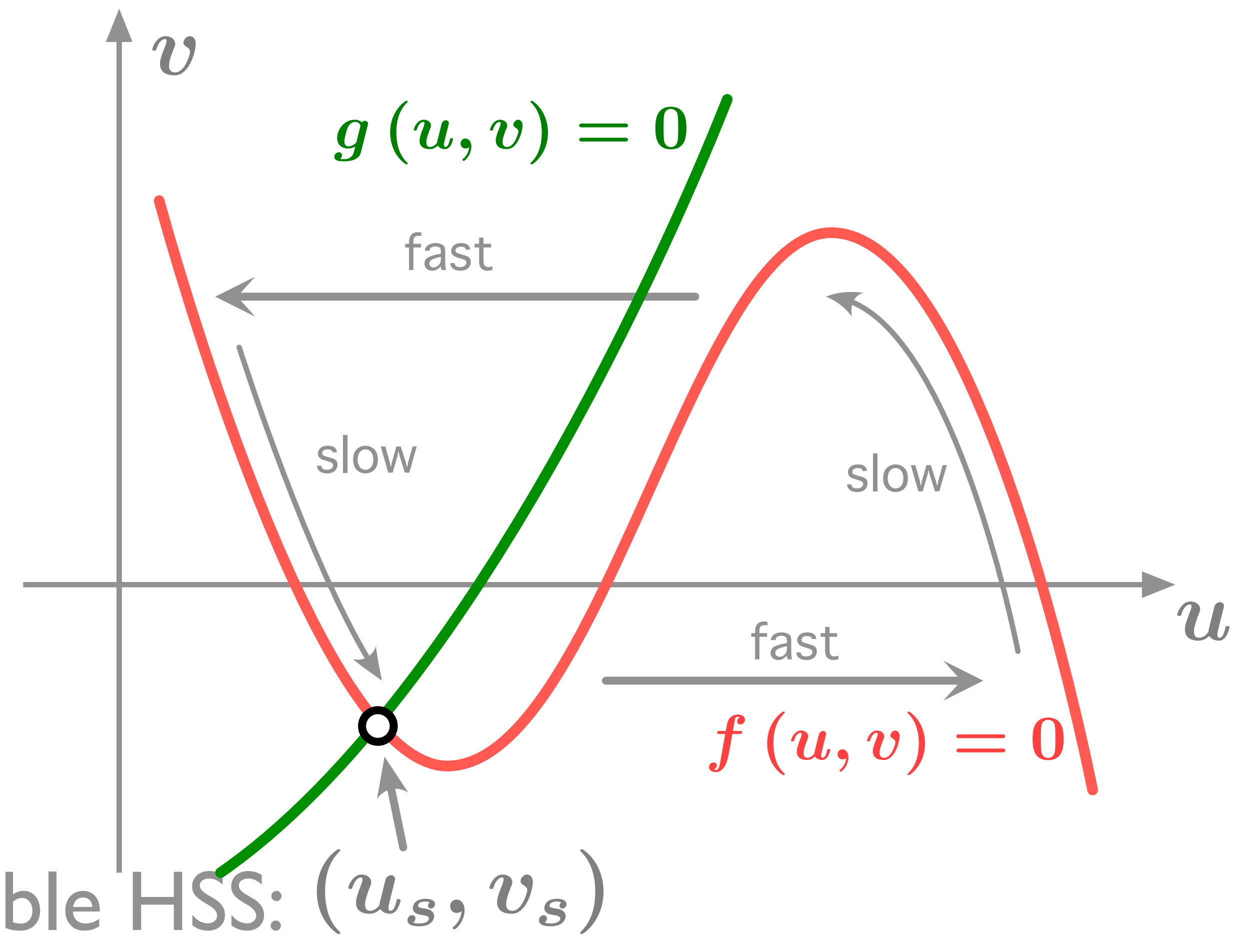
Computer simulation

Excitable media

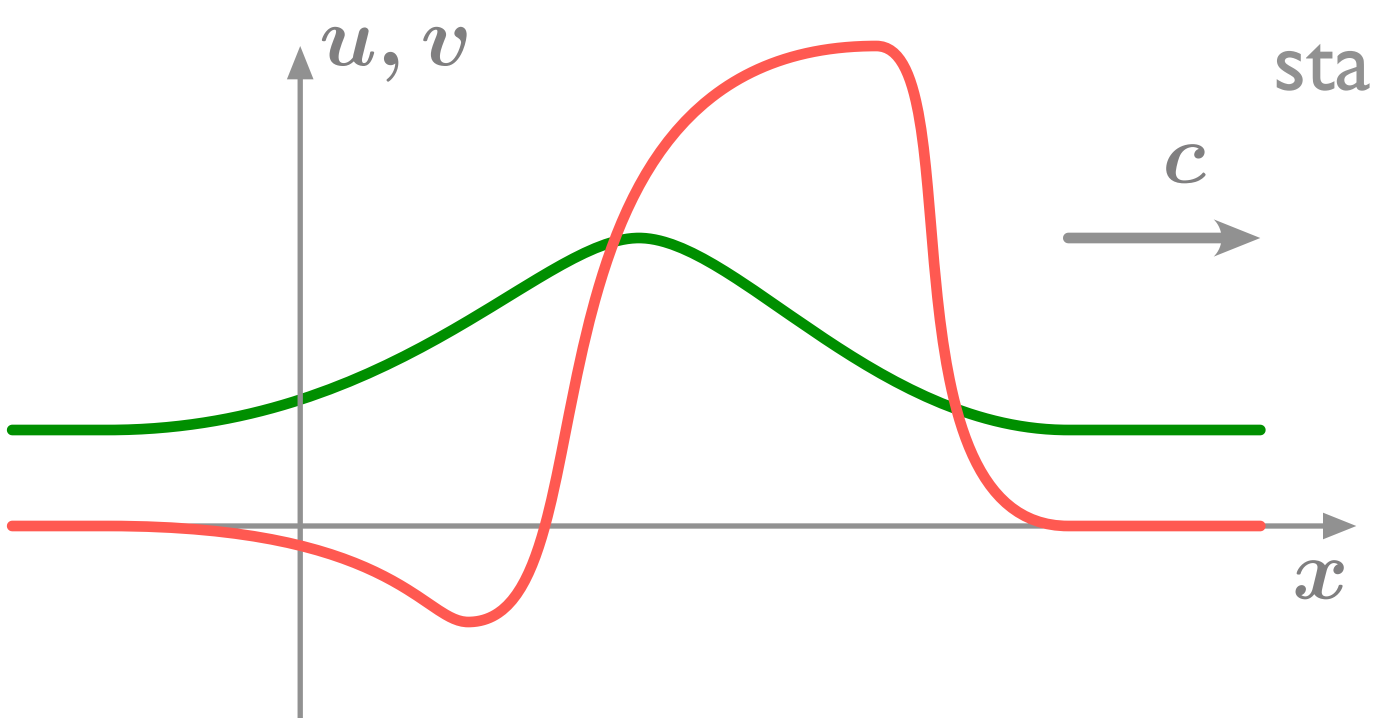
$$u_t = \frac{1}{\epsilon} f(u, v) + u_{xx}, \quad \epsilon \to 0$$

$$v_t = g(u, v) + \delta v_{xx}$$

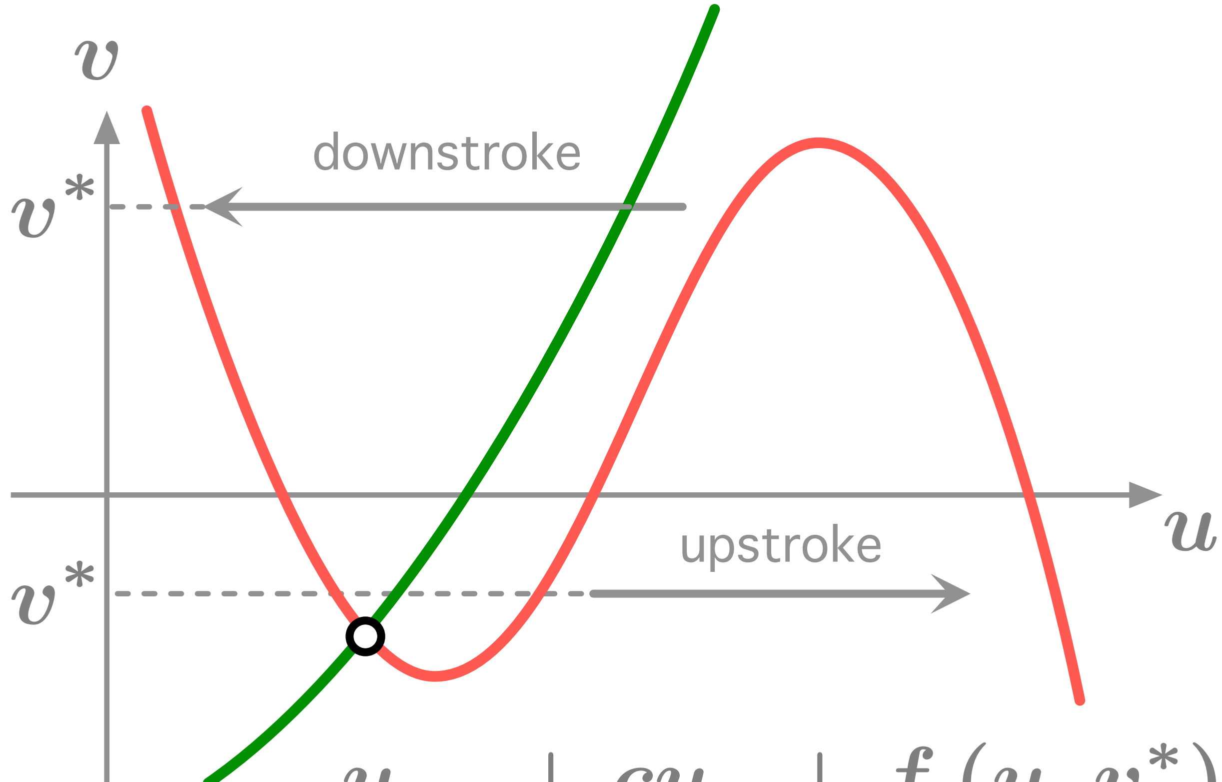
u: 1/ϵ-fast activator variable
v: slow inhibitor variable



$$f(u_s, v_s) = 0,$$
$$g(u_s, v_s) = 0$$

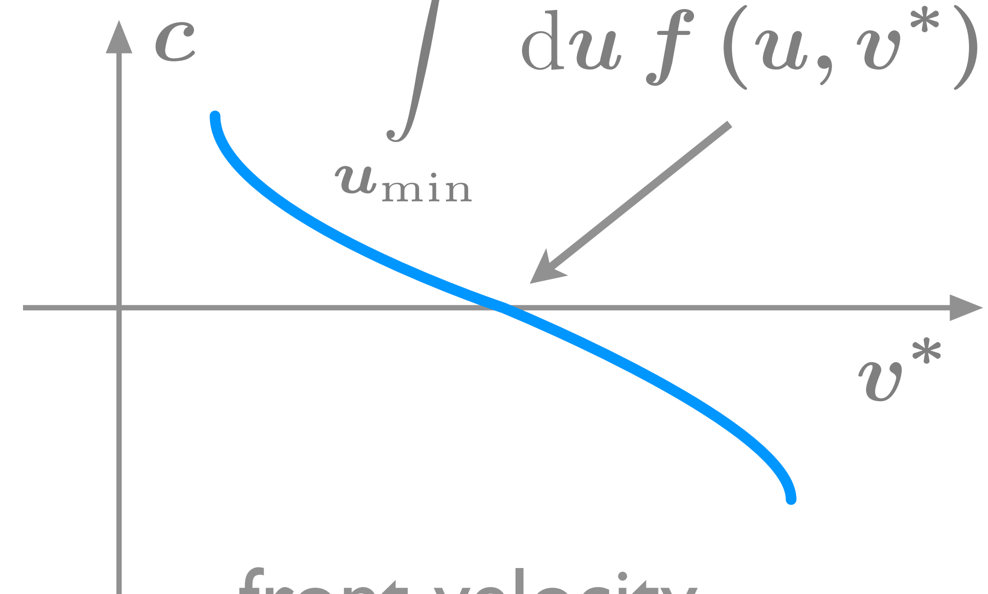


Upstroke and downstroke

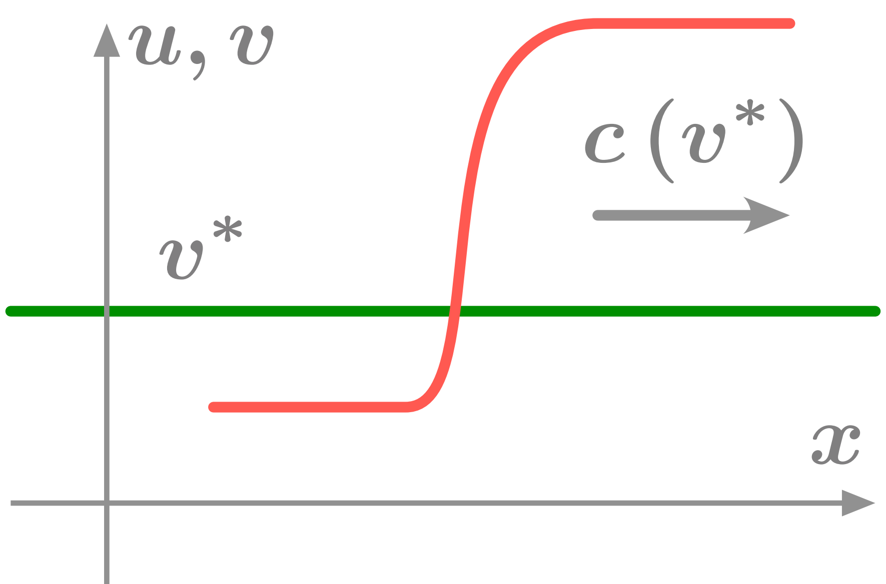


$$u_{xx} + cu_x + f\left(u, v^*\right) = 0$$

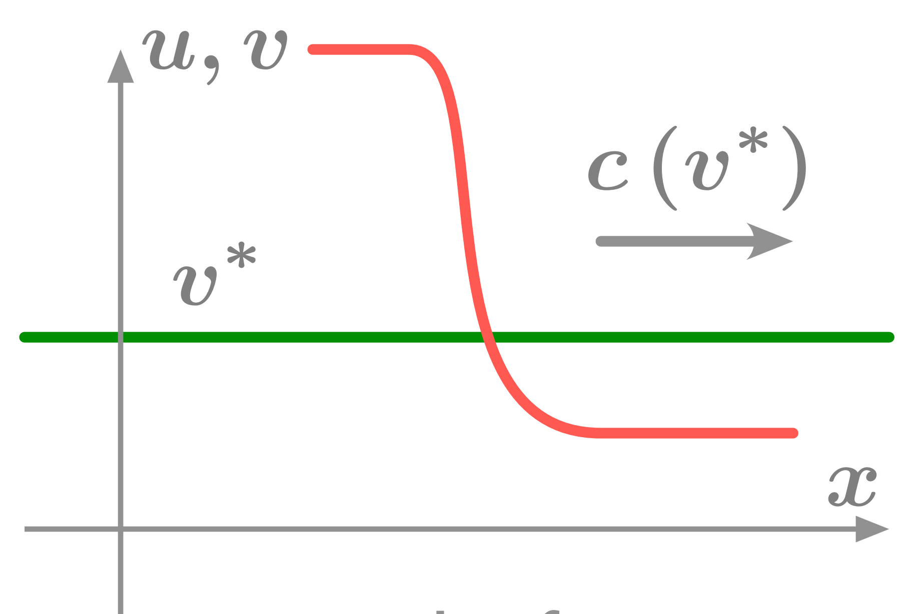
$$\int_{u_{\min}}^{u_{\max}} \mathrm{d}u\, f\left(u, v^*\right) = 0$$



front velocity
as function of v

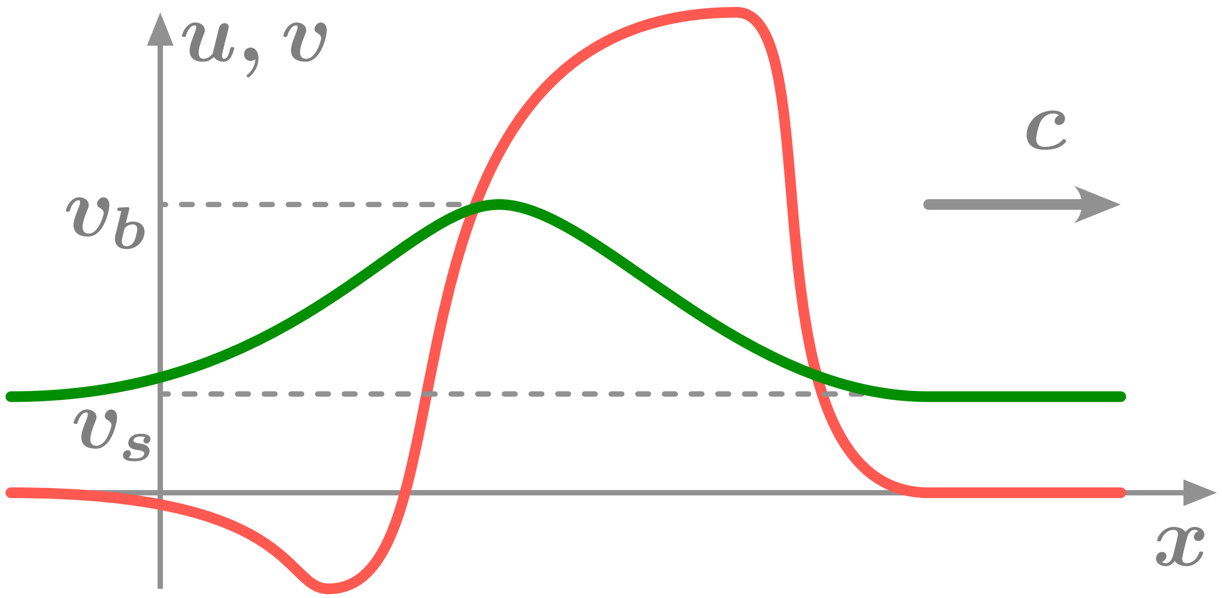


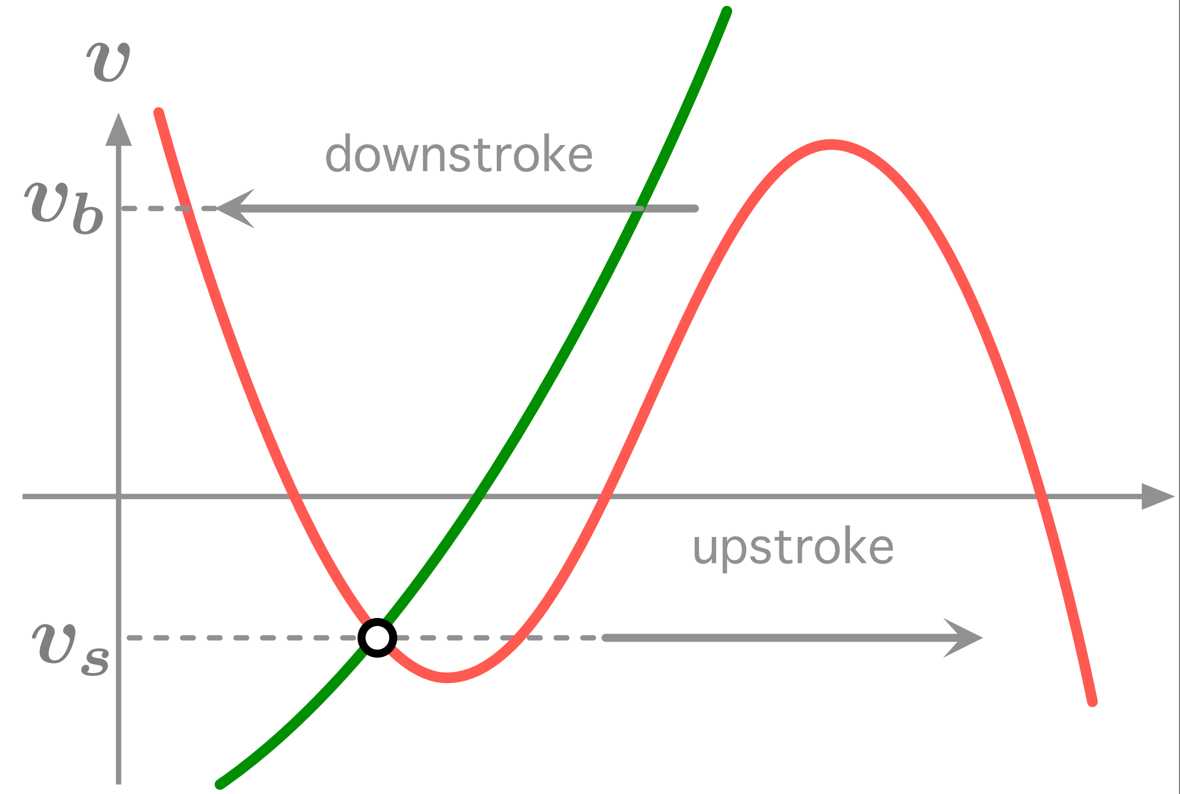
downstroke front



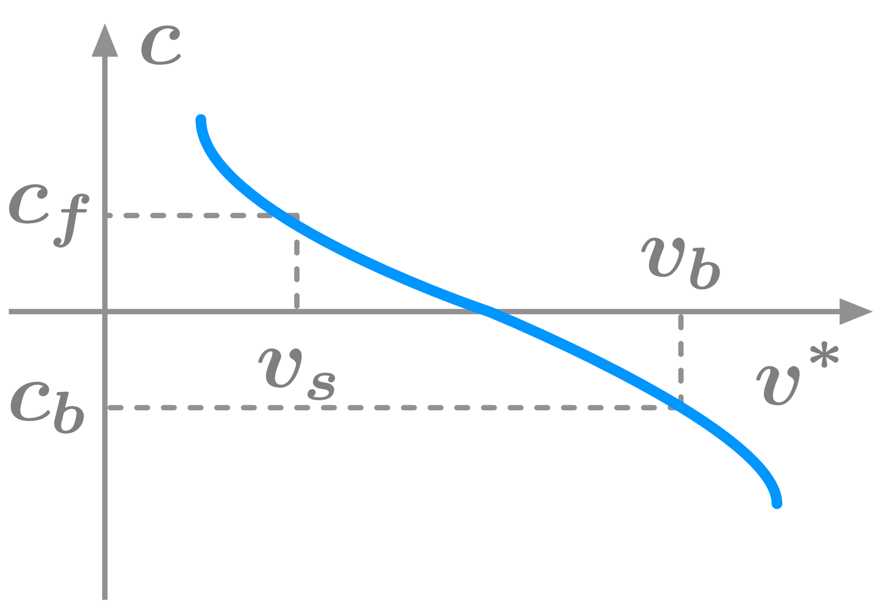
upstroke front

Solitary pulse





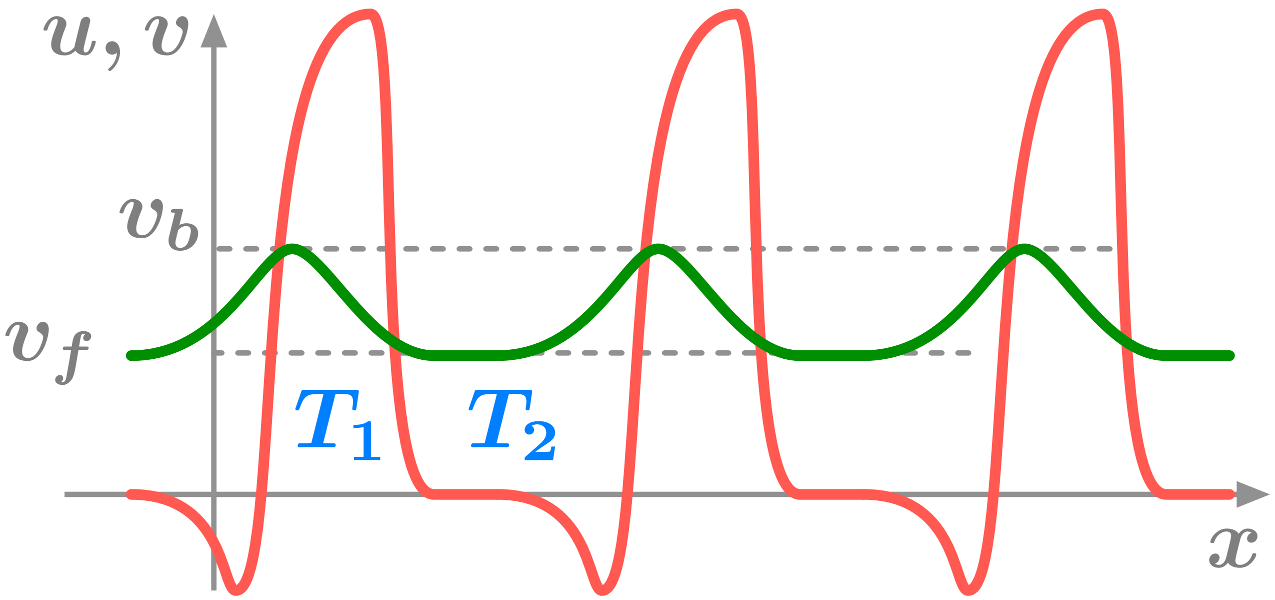
upstroke is exactly at v_s

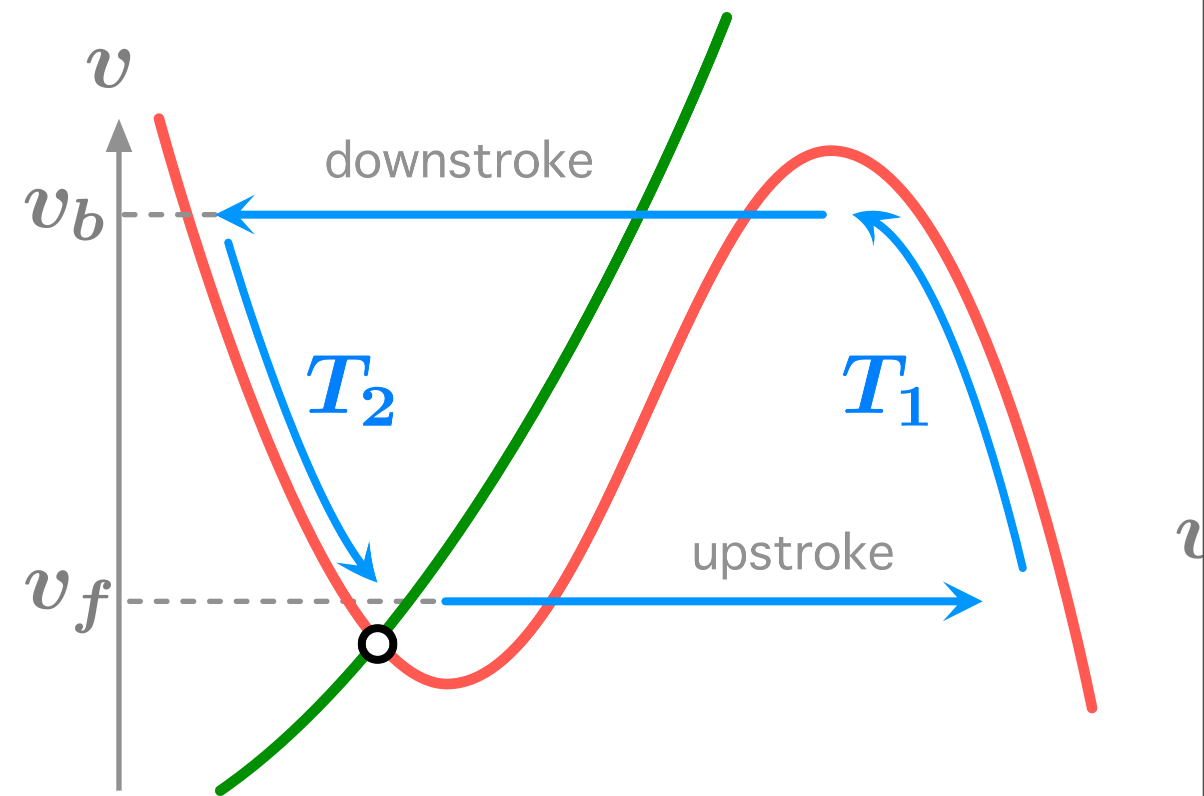


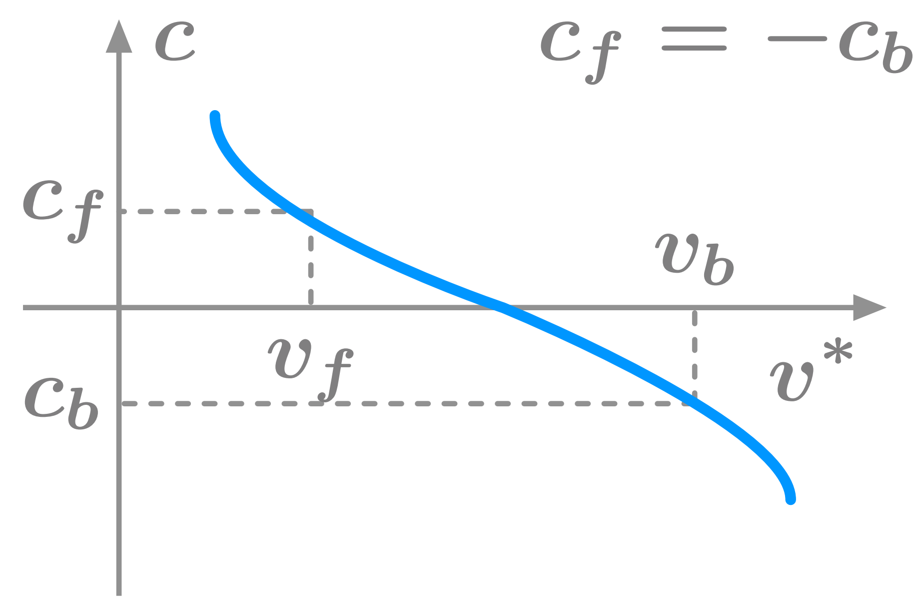
$$c_f = -c_b$$

how the *v* value of the pulse back is chosen

Periodic pulse trains



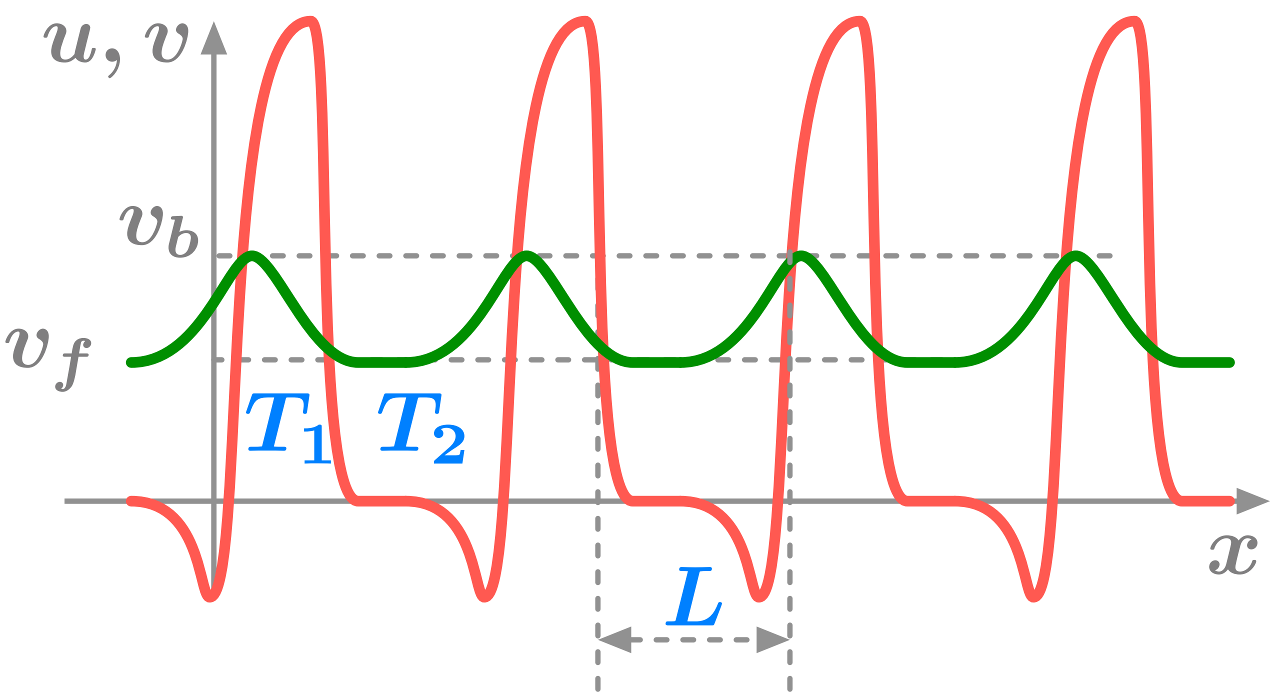


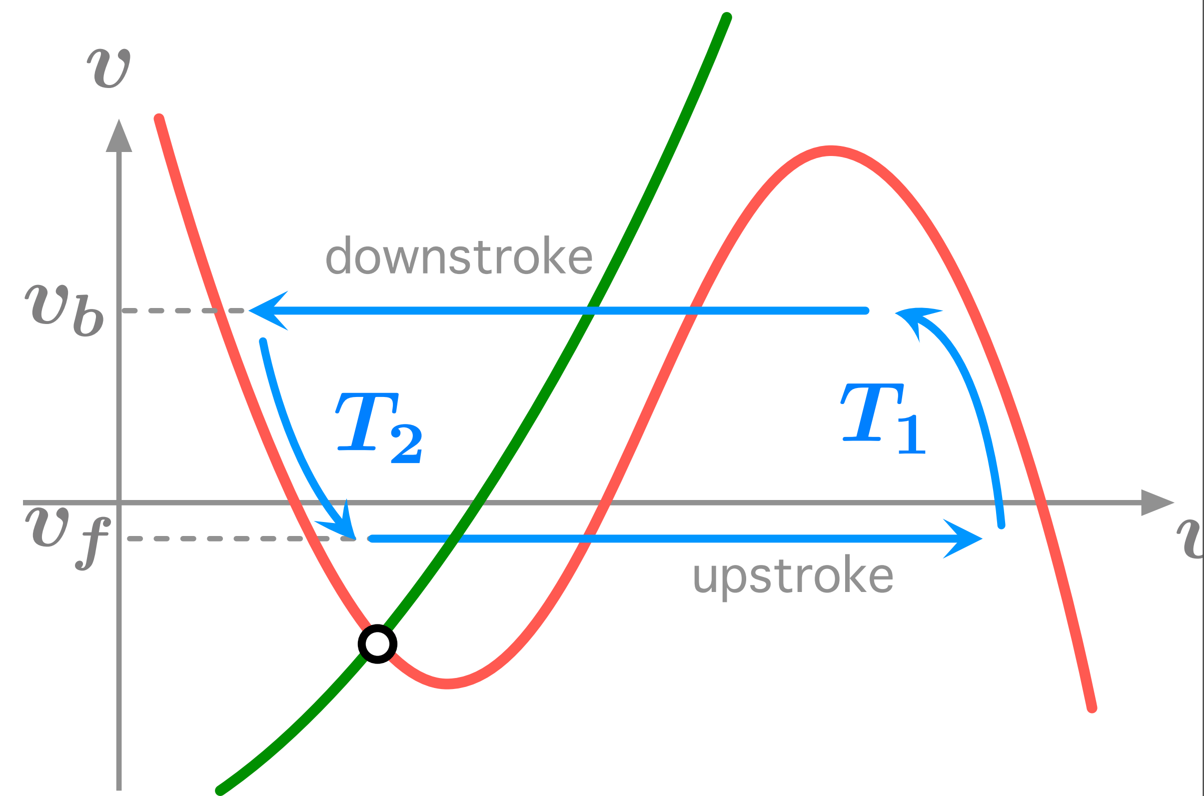


$$c_f = -c_b$$

Period of wave train determines its speed

Periodic pulse trains

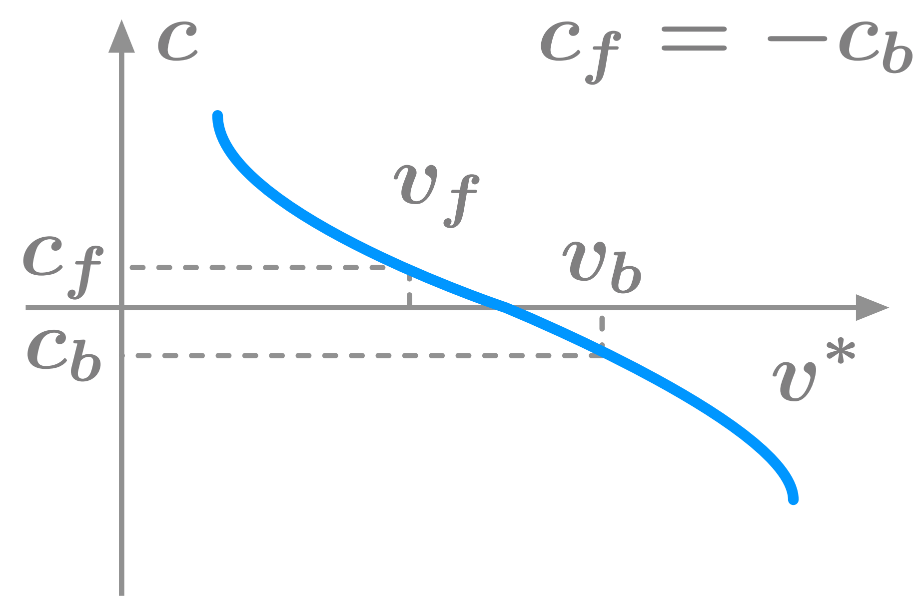




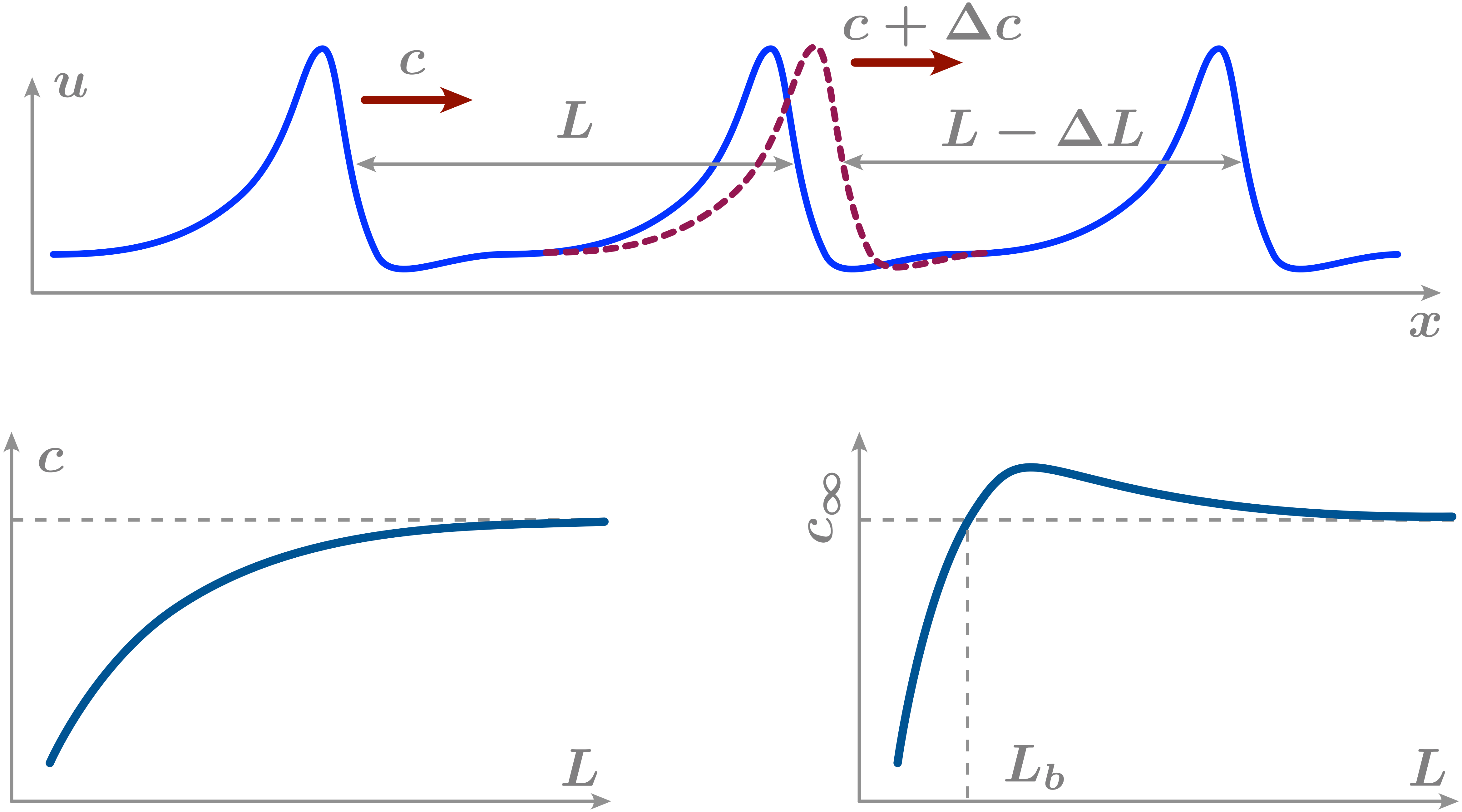
Period of wave train determines its speed

(nonlinear) dispersion:

$$c = c(T) \quad \text{or} \quad c = c(L)$$



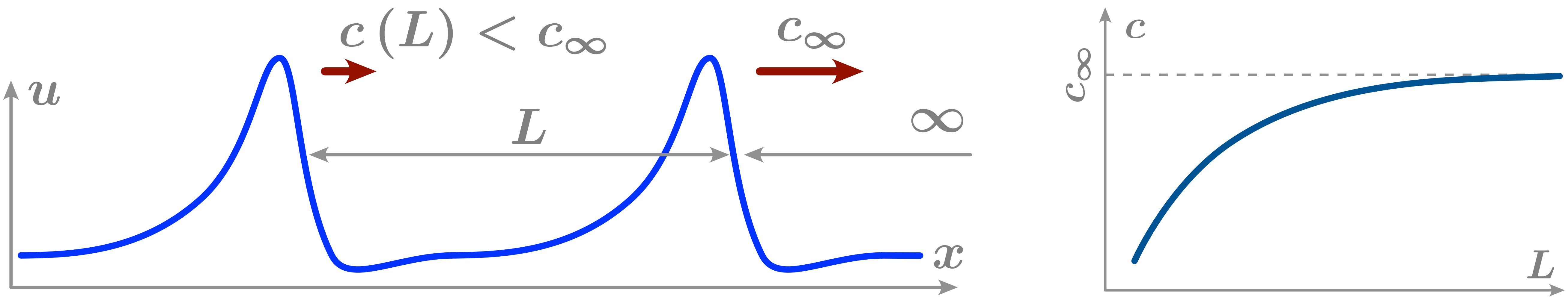
Dispersion of Pulse Trains



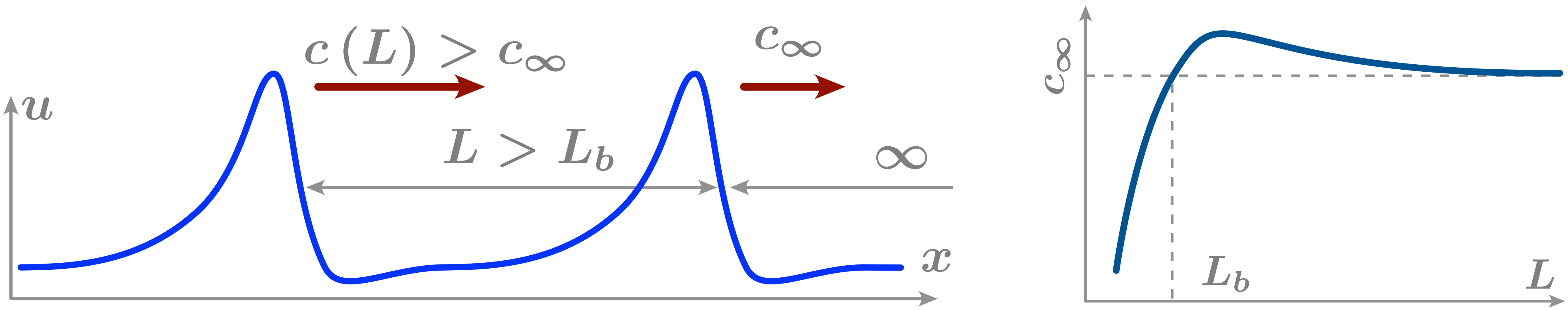
normal dispersion

anomalous dispersion

Interaction of pulses



case of **normal dispersion**



anomalous dispersion, bound states possible!

Overview

- One-component bistable RDS - connecting fronts, unique front profile and propagation speed
- Two-component RDS - excitability pulses which consist of up- and downstroke, also unique profile/speed for solitary pulses
- Pulses are obtained by connecting two fronts together and exploring time-scale separation
- Periodic pulse trains are characterized by dispersion