Probability Density and Expectations

For a probability density function (pdf) $p_x$ of a random variable $x$, the expectation of a function $g(x)$ is denoted by

$$E\{g(x)\} := \int_{-\infty}^{+\infty} g(\xi)p_x(\xi)\,d\xi.$$ 

We define the mean, second moment, and the variance of $x$ as $m_x = E\{x\}$, $r_{xx} = E\{x^2\}$, and $\sigma^2 = E\{(x-m_x)^2\}$, respectively. Probability densities are always normalized: $E\{1\} = 1$. The generalization to random vectors $\mathbf{x}$ is provided below.

1. **Linearity of Expectations**

Let $x_i (i = 1, \ldots, m)$ be a set of different, but not independent, random variables, i.e. the vector $\mathbf{x} := (x_1, \ldots, x_m)^T$ is random with pdf $p_{\mathbf{x}}$. Prove that the expectations satisfy the linearity property

$$E\left\{ \sum_{i=1}^{m} a_i x_i \right\} = \sum_{i=1}^{m} a_i E\{x_i\},$$

where the $a_i$ are arbitrary scalar coefficients.

2. **Moments**

Explicitly compute the mean, second moment, and variance of a random variable distributed uniformly in the interval $[a, b]$ for $b > a$.

Joint and Marginal Densities

The marginal densities $p_x(\xi)$ and $p_y(\eta)$ of the random variables $x$ and $y$ are obtained from their joint density $p_{x,y}(\xi, \eta)$ through integration:

$$p_x(\xi) = \int_{-\infty}^{\infty} p_{x,y}(\xi, \eta)\,d\eta,$$
$$p_y(\eta) = \int_{-\infty}^{\infty} p_{x,y}(\xi, \eta)\,d\xi.$$

3. **Marginal Density**

Calculate the marginal densities of

$$p_{x,y}(\xi, \eta) = \begin{cases} \frac{1}{2}(2 - \xi)(\xi + \eta) & \text{for } \xi \in [0, 2], \eta \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$
Uncorrelatedness and Independence

Two scalar random variables $x$ and $y$ are *uncorrelated* if their covariance $c_{xy}$ is zero. The covariance is defined as

$$c_{xy} := E[(x - m_x)(y - m_y)].$$

The random variables $x$ and $y$ are said to be *independent* if and only if their joint density factorizes into the product of their marginal densities,

$$p_{x,y}(\xi, \eta) = p_x(\xi) p_y(\eta)$$

for all $\xi$ and for all $\eta$.

4. Independence

Show that independent random variables $x$ and $y$ satisfy the basic property

$$E\{g(x) h(y)\} = E\{g(x)\} E\{h(y)\}$$

where $g(x)$ and $h(y)$ are absolutely integrable functions. Use the generalization of expectations to two random variables,

$$E\{g(x) h(y)\} = \int d\xi \int d\eta p_{x,y}(\xi, \eta) g(\xi) h(\eta).$$

5. Independence Implies Uncorrelatedness

a) Prove that independence of $x$ and $y$ implies uncorrelatedness.

b) Give an example to show that uncorrelatedness does *not* imply independence. For example, assume that the pair $(x, y)$ takes on discrete values $(0, 1), (0, -1), (1, 0), (-1, 0)$ with probability $1/4$ each.

6. Not Independent

Argue that the random variables $x$ and $y$ in Problem 3 are not independent.

7. Orthonormal Transformations (Rotations)

Assume that $x_1$ and $x_2$ are zero-mean, correlated random variables. Any orthonormal transformation of $x_1$ and $x_2$ can be represented in the form

$$y_1 = +x_1 \cos \alpha + x_2 \sin \alpha$$
$$y_2 = -x_1 \sin \alpha + x_2 \cos \alpha$$

where the parameter $\alpha$ defines a rotation angle of coordinate axes. Let the variances be $E\{x_1^2\} = \sigma_1^2 > 0$, $E\{x_2^2\} = \sigma_2^2 > 0$, and the covariance be $E\{x_1 x_2\} = \rho \sigma_1 \sigma_2$. Find the angle $\alpha$ for which $y_1$ and $y_2$ become uncorrelated.