Humboldt-Universität zu Berlin
Institute for Theoretical Biology

Exercise Set 1
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Computational Neuroscience IV: Analysis of Neural Systems

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Probability Density and Expectations

For a probability density function (pdf) p_x of a random variable x, the expectation of a function $g(x)$ is denoted by

$$E\{g(x)\} := \int_{-\infty}^{+\infty} g(\xi)\, p_x(\xi)\, \mathrm{d}\xi \ .$$

We define the mean, second moment, and the variance of x as $m_x = E\{x\}$, $r_{xx} = E\{x^2\}$, and $\sigma^2 = E\{(x - m_x)^2\}$, respectively. Probability densities are always normalized: $E\{1\} = 1$. The generalization to random vectors $\mathbf{x}$ is provided below.

1. Linearity of Expectations
Let x_i $(i = 1, \ldots, m)$ be a set of different, but not independent, random variables, i.e. the vector $\mathbf{x} := (x_1, \ldots, x_m)^T$ is random with pdf $p_\mathbf{x}$. Prove that the expectations satisfy the linearity property

$$E\left\{\sum_{i=1}^{m} a_i\, x_i\right\} = \sum_{i=1}^{m} a_i\, E\{x_i\} \ ,$$

where the a_i are arbitrary scalar coefficients.

2. Moments
Explicitly compute the mean, second moment, and variance of a random variable distributed uniformly in the interval $[a, b]$ for $b > a$.

Joint and Marginal Densities

The marginal densities $p_x(\xi)$ and $p_y(\eta)$ of the random variables x and y are obtained from their joint density $p_{x,y}(\xi, \eta)$ through integration:

$$p_x(\xi) = \int_{-\infty}^{\infty} p_{x,y}(\xi, \eta)\, \mathrm{d}\eta \ , \qquad p_y(\eta) = \int_{-\infty}^{\infty} p_{x,y}(\xi, \eta)\, \mathrm{d}\xi \ .$$

3. Marginal Density
Calculate the marginal densities of

$$p_{x,y}(\xi, \eta) = \begin{cases} \frac{3}{7}(2 - \xi)(\xi + \eta) & \text{for} \quad \xi \in [0, 2], \eta \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Uncorrelatedness and Independence

Two scalar random variables x and y are *uncorrelated* if their covariance c_{xy} is zero. The covariance is defined as

$$c_{xy} := E\{(x - m_x)(y - m_y)\} \ .$$

The random variables x and y are said to be *independent* if and only if their joint density factorizes into the product of their marginal densities,

$$p_{x,y}(\xi, \eta) = p_x(\xi)\, p_y(\eta)$$

for all ξ and for all η.

4. Independence

Show that independent random variables x and y satisfy the basic property

$$E\{g(x)\, h(y)\} = E\{g(x)\}\, E\{h(y)\}$$

where $g(x)$ and $h(y)$ are absolutely integrable functions. Use the generalization of expectations to two random variables, $E\{g(x)\, h(y)\} = \int \mathrm{d}\xi \int \mathrm{d}\eta\, p_{x,y}(\xi, \eta)\, g(\xi)\, h(\eta)$.

5. Independence Implies Uncorrelatedness

a) Prove that independence of x and y implies uncorrelatedness.

b) Give an example to show that uncorrelatedness does *not* imply independence. For example, assume that the pair (x, y) takes on discrete values $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$ with probability $1/4$ each.

6. Not Independent

Argue that the random variables x and y in Problem 3 are not independent.

7. Orthonormal Transformations (Rotations)

Assume that x_1 and x_2 are zero-mean, correlated random variables. Any orthonormal transformation of x_1 and x_2 can be represented in the form

$$\begin{aligned} y_1 &= +x_1 \cos\alpha + x_2 \sin\alpha \\ y_2 &= -x_1 \sin\alpha + x_2 \cos\alpha \end{aligned}$$

where the parameter α defines a rotation angle of coordinate axes. Let the variances be $E\{x_1^2\} = \sigma_1^2 > 0$, $E\{x_2^2\} = \sigma_2^2 > 0$, and the covariance be $E\{x_1\, x_2\} = \rho\, \sigma_1\, \sigma_2$. Find the angle α for which y_1 and y_2 become uncorrelated.

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Problems handed out on Monday, 24.04.2006.
Solutions to be handed in by Friday, 05.05.2006, 8^{30} am.
Discussion of the problems on Friday, 05.05.2006, $8^{30}-10^{00}$ am in front of room 2317 I-W (Zwischengeschoß).