Humboldt-Universität zu Berlin
Institute for Theoretical Biology

Exercise Set 2
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Computational Neuroscience IV: Analysis of Neural Systems

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Mean Vector, Correlation, and Covariance Matrix

The first moment of an n-dimensional random vector $\mathbf{x}$ with pdf $p_{\mathbf{x}}$ is called the mean vector $\mathbf{m}_{\mathbf{x}}$. It is defined as the expectation of $\mathbf{x}$,

$$\mathbf{m}_{\mathbf{x}} = E\{\mathbf{x}\} = \int p_{\mathbf{x}}(\boldsymbol{\xi})\, \boldsymbol{\xi}\, \mathrm{d}\boldsymbol{\xi} \ ,$$

where each component m_{x_i} of $\mathbf{m}_{\mathbf{x}}$ is given by $m_{x_i} = E\{x_i\} = \int_{-\infty}^{+\infty} p_{x_i}(\xi)\, \xi\, \mathrm{d}\xi$ for $i = 1, \ldots, n$.

The second moment of $\mathbf{x}$ is called the $n \times n$ correlation matrix

$$\mathbf{R}_{\mathbf{x}} = E\{\mathbf{x}\,\mathbf{x}^T\} \ ,$$

where the correlation r_{ij} between the ith and jth component of $\mathbf{x}$ is given by $r_{ij} = \int \mathrm{d}\xi \int \mathrm{d}\eta\, p_{x_i,x_j}(\xi, \eta)$.

The second *central* moment of $\mathbf{x}$ is the covariance matrix

$$\mathbf{C}_{\mathbf{x}} = E\{(\mathbf{x} - \mathbf{m}_{\mathbf{x}})\,(\mathbf{x} - \mathbf{m}_{\mathbf{x}})^T\} \ .$$

1. Show that $\mathbf{C}_{\mathbf{x}}$ and $\mathbf{R}_{\mathbf{x}}$ satisfy the relation $\mathbf{R}_{\mathbf{x}} = \mathbf{C}_{\mathbf{x}} + \mathbf{m}_{\mathbf{x}}\,\mathbf{m}_{\mathbf{x}}{}^T$.

2. Consider an n-dim random vector $\mathbf{x}$ with diagonal covariance matrix $\mathbf{C}_{\mathbf{x}} = \mathrm{diag}\{\sigma_{x_1}^2, \sigma_{x_2}^2, \ldots, \sigma_{x_n}^2\}$. Show that the variance σ_y^2 of the sum $y = \sum_{i=1}^n x_i$ equals $\mathrm{tr}\{\mathbf{C}_{\mathbf{x}}\}$, where $\mathrm{tr}\{\mathbf{C}_{\mathbf{x}}\} := \sum_{i=1}^n E\{(x_i - m_{x_i})^2\}$.

3. Show that two (zero-mean) random variables that have a jointly gaussian distribution are statistically independent if and only if they are uncorrelated. (Hint: the gaussian pdf in the n-dimensional case is

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}(\det\, \mathbf{C}_{\mathbf{x}})^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m}_{\mathbf{x}})^T \mathbf{C}_{\mathbf{x}}^{-1}(\mathbf{x} - \mathbf{m}_{\mathbf{x}})\right) \ .$$

Uncorrelatedness means that the matrix $\mathbf{C}_{\mathbf{x}}$ is diagonal and hence nonsingular. Show that this implies that the joint pdf can be factorized.)

Eigenvectors and Eigenvalues

An eigenvector $\mathbf{w}$ of a square matrix $\mathbf{A}$ is a non-zero vector that satisfies

$$\mathbf{A}\,\mathbf{w} = \lambda\,\mathbf{w}$$

where λ is the eigenvalue associated with that eigenvector. For a real, symmetric ($\mathbf{A} = \mathbf{A}^T$), and positive semi-definite matrix ($\mathbf{x}^T \mathbf{A}\,\mathbf{x} \geq 0$ for all n-vectors $\mathbf{x} \neq 0$, which implies det $\mathbf{A} \geq 0$), such as a correlation matrix, eigenvalues are real and non-negative, and eigenvectors associated with different eigenvalues are mutually orthogonal.

4. Consider a two-dimensional gaussian random vector $\mathbf{x}$ with mean $\mathbf{m_x} = (2,1)^T$ and covariance matrix

$$\mathbf{C_x} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} .$$

Find the eigenvalues and normalized eigenvectors of $\mathbf{C_x}$. Show that the eigenvectors are orthogonal. Indicate the shape of the gaussian density through a contour line, and plot $\mathbf{m_x}$ as well as the eigenvectors.

5. Reconsider problem 7 from exercise sheet 1, where we had $\mathbf{C_x} = [(\sigma_1^2, \rho\,\sigma_1\sigma_2)^T, (\rho\,\sigma_1\sigma_2, \sigma_2^2)^T]$:

a) Calculate the eigenvalues of $\mathbf{C_x}$.

b) Derive the allowed range for the so-called 'correlation coefficient' ρ using the fact that $\mathbf{C_x}$ has non-negative eigenvalues? (Answer: $-1 \leq \rho \leq 1$)

c) Calculate the eigenvalues and eigenvectors of $\mathbf{C_x}$ in the special case $\sigma_1 = \sigma_2 = 1$, and indicate the shape of a gaussian pdf for $\rho \in \{-1, -0.9, 0, 0.5\}$.

6. What conditions should the elements of the matrix

$$\mathbf{R_x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfy so that $\mathbf{R_x}$ is a valid correlation matrix of a two-dimensional random vector $\mathbf{x}$?

7. Compute a whitening transformation for the random vector $\mathbf{x}$ with covariance matrix

$$\mathbf{C_x} = \begin{bmatrix} 2.5 & 1.5 \\ 1.5 & 2.5 \end{bmatrix} .$$

Higher-Order Moments

The jth moment α_j of a scalar random variable x with pdf $p_x(\xi)$ is defined by $\alpha_j = E\{x^j\}$. The jth central moment μ_j of x is

$$\mu_j = E\{(x-\alpha_1)^j\} = \int_{-\infty}^{+\infty} \mathrm{d}\xi\,(\xi-\alpha_1)^j\,p_x(\xi), \quad \text{for } j = 1,2,\ldots$$

8. The third central moment μ_3 is called skewness. Show that the skewness of a random variable having symmetric pdf is zero.

9. Related to the fourth central moment μ_4 is the kurtosis, which is defined in the zero-mean case by

$$\mathrm{kurt}(x) = E\{x^4\} - 3[E\{x^2\}]^2 .$$

Show that the kurtosis of a gaussian random variable is zero. Hint: use $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x^2/(2\sigma^2)} = -x\mathrm{e}^{-x^2/(2\sigma^2)}/\sigma^2$ and perform a partial integration in the calculation of α_4.

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Problems handed out on Monday, 24.04.2006.
Solutions to be handed in by Monday, 08.05.2006, 12^{15} pm
Discussion and presentation of the problems on Friday, 12.05.2006, $8^{30} - 10^{00}$ am in front of room 2317 I-W (Zwischengeschoß).